

Remark on Koide's Z_3 -symmetric parametrization of quark masses

Piotr Żenczykowski *

Division of Theoretical Physics

the Henryk Niewodniczański Institute of Nuclear Physics

Polish Academy of Sciences

Radzikowskiego 152, 31-342 Kraków, Poland

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Abstract

The charged lepton masses may be parametrized in a Z_3 -symmetric language appropriate to the discussions of Koide's formula. The phase parameter δ_L appearing in this parametrization is experimentally indistinguishable from $2/9$. We analyse Koide's parametrization for the up (U) and down (D) quarks and argue that the data are suggestive of the low-energy values $\delta_U = \delta_L/3 = 2/27$ and $\delta_D = 2\delta_L/3 = 4/27$.

*E-mail: piotr.zenczykowski@ifj.edu.pl

Although the Standard Model is extremely successful, various questions concerning elementary particles cannot be answered within it. Among them, notwithstanding the recent discovery at CERN, there is still the issue of particle masses. This problem seems to be intimately related to the appearance of three generations of leptons and quarks. Since the understanding of these issues may require completely new ideas, phenomenological identification of regularities observed in the pattern of particle masses and mixings is of crucial importance. It may provide us with analogues of Balmer and Rydberg's formulae and should hopefully lead us to a genuinely new physics.

1. One of the most interesting of such regularities was discovered by Koide [1] (for a brief review see [2]). It reads:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{1 + k_L^2}{3}, \quad (1)$$

with k_L equal exactly 1.

In fact, if one plugs into the above formula the central values of experimental electron and muon masses [3]:

$$\begin{aligned} m_e(exp) &= 0.510998928(11) \text{ MeV}, \\ m_\mu(exp) &= 105.65836715(35) \text{ MeV}, \end{aligned} \quad (2)$$

one finds (with $k_L = 1$) that the larger of the solutions of Eq. (1) is

$$m_\tau = 1776.9689 \text{ MeV}, \quad (3)$$

to be compared with the experimental τ mass

$$m_\tau(exp) = 1776.82 \pm 0.16 \text{ MeV}. \quad (4)$$

Discussions of this success of Koide's formula are often formulated in a Z_3 -symmetric language by parametrizing the masses of any three given fermions f_1, f_2, f_3 as [4, 5]:

$$\sqrt{m_j} = \sqrt{M_f} \left(1 + \sqrt{2} k_f \cos \left(\frac{2\pi j}{3} + \delta_f \right) \right), \quad (j = 1, 2, 3). \quad (5)$$

In general, with appropriately chosen three parameters (here: the overall mass scale M_f and the pattern parameters k_f and δ_f) one can fit any three-particle spectrum, of course. The above choice of parametrization is, however, particularly suited to Koide's formula, since the latter is then independent of parameter δ_f , as specified in Eq. (1). That the resulting formula works then for k_L equal exactly 1 (or almost 1 with very high precision) is truly amazing.

2. The peculiar feature of formula (1) is that it works at a low energy scale and not at some high mass scale (like M_Z or 10^{16} GeV) [6, 7, 8]. For example, taking the values of charged lepton masses as appropriate at the scale of M_Z ($m_e = 0.486755106$ MeV, $m_\mu = 102.740394$ MeV, $m_\tau = 1746.56$ MeV) one finds that $k_L(M_Z) = 1.00188$, i.e. it deviates from unity quite significantly. If that value of k_L worked for physical masses it would remove much of the excitement Koide's formula generates.

Attempts have been made to apply Koide's formula to quarks and neutrinos. The general conclusion is that the formula does not work there (i.e. $k_f \neq 1$). Specifically, using the quark mass values as appropriate at $\mu = 2$ GeV, for the down quarks (D) one obtains the values of k_D around 1.08, while for the up quarks (U) one gets k_U around 1.25 [7, 9, 10] (with the mathematically allowed region $0 \leq k_f \leq \sqrt{2}$). Furthermore, for neutrinos ν one estimates directly from experiment that $k_\nu \leq 0.81$ [10]. If a higher energy scale μ is taken, the agreement deteriorates further (at the M_Z mass scale one gets $k_D = 1.12$ and $k_U = 1.29$).

3. Given the success of Koide's parametrization (5) with $k_f = 1$ for charged leptons ($f = L$) and its failure for other fundamental fermions ($f = U, D, \nu$), one should perhaps look in a different direction. In fact, parametrization (5) reveals another miracle in the charged lepton sector. Namely, using the experimental values of charged lepton masses one can determine the value of the phase parameter δ_L .

From Eq. (5) one finds

$$\frac{1}{\sqrt{2}}(\sqrt{m_2} - \sqrt{m_1}) = \sqrt{M_f} k_f \sqrt{3} \sin \delta_f, \quad (6)$$

$$\frac{1}{\sqrt{6}}(2\sqrt{m_3} - \sqrt{m_2} - \sqrt{m_1}) = \sqrt{M_f} k_f \sqrt{3} \cos \delta_f. \quad (7)$$

Since δ_f is free we may assume $m_1 \leq m_2 \leq m_3$ without any loss of generality. Then, independently of the value of k_L , one gets

$$\tan \delta_L = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}. \quad (8)$$

From the experimental values of Eqs (2, 4) one then calculates:

$$\delta_L = 0.2222324, \quad (9)$$

which is extremely close to $\delta_L = 2/9$ [11, 12]. After inverting formula (8) and assuming $\delta_L = 2/9$ one can predict the value of τ mass, given the experimental masses of electron and muon. The result is:

$$m_\tau = 1776.9664 \text{ MeV}. \quad (10)$$

This is as good a prediction of the tauon mass as that given by the original Koide's formula (3). The two predictions of Eqs (3,10) are mutually incompatible, but either

of them leads to an excellent prediction for m_τ . They could be made consistent with each other by allowing extremely tiny departures of either δ_L from $2/9$ or k_L from 1. Keeping in mind the violation of Koide's formula for quarks and neutrinos, one should perhaps try the $\delta_L = 2/9$ alternative, for example by maintaining the values of all δ_f equal to δ_L . An attempt in a similar direction was made by Rosen [12]. He keeps the values of all δ_f equal to $2/9$ and the values of all k_f at 1.¹ Then, he constructs a Z_3 -symmetric model which modifies (in a calculable way) the value of M_q for each quark q_j separately (hence $M_q \rightarrow M_{q,j}$). However, then the Koide parametrization of Eq.(5) ceases to be valid.

4. On the other hand, a different route possible may be taken which keeps parametrization (5) intact. Namely, one could refrain for the time being from the discussions of Koide-like formulas (with $k_f \neq 1$). Instead, one might try to analyze the issue of δ_f in more detail. After all, the assumption of $\delta_L = 2/9$ yields as good a prediction for the tauon mass as the assumption of $k_L = 1$.

Thus, the idea is to analyse what the experimental values of quark masses tell us about the δ_f parameters for $f = U, D$. For this simple exercise we take the following typical set of the values of experimental masses at $\mu = 2$ GeV (in MeV):

$$\begin{aligned} m_u &= 1.7 - 3.3, \\ m_c &= 1270^{+70}_{-90}, \\ m_t &= 172000 \pm 1600, \\ m_d &= 4.1 - 5.8, \\ m_s &= 101^{+29}_{-21}, \\ m_b &= 4190^{+180}_{-60}. \end{aligned} \tag{11}$$

For the discussion of δ_f we introduce $z_k = \sqrt{m_k/m_3}$ (assuming $m_1 < m_2 < m_3$). Thus

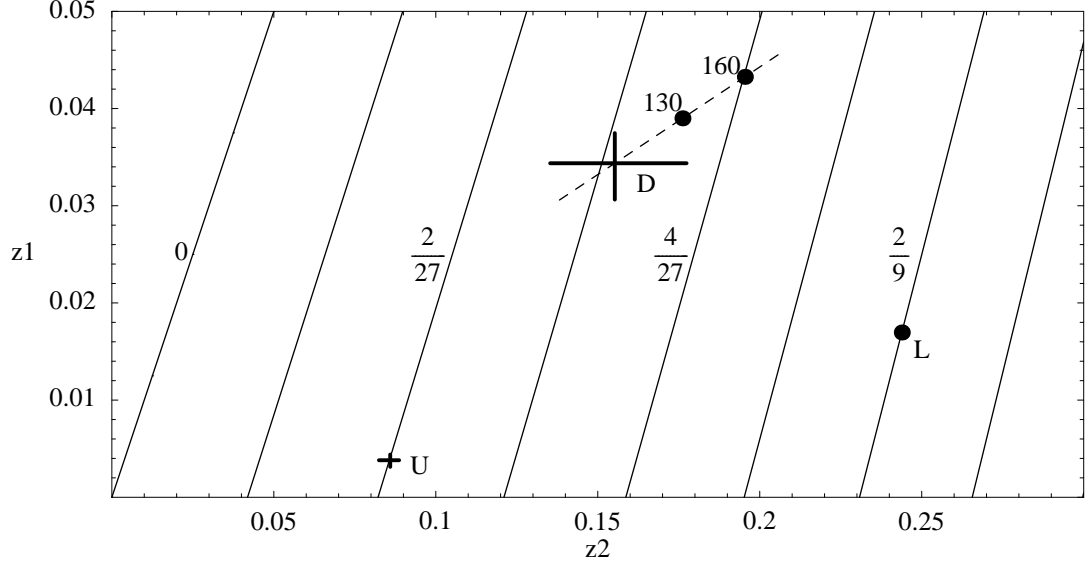
$$\delta_f = \arctan \left(\sqrt{3} \frac{z_2 - z_1}{2 - z_2 - z_1} \right). \tag{12}$$

Fig. 1 shows a contour plot of $\delta_f(z_1, z_2)$ and the corresponding approximate positions of (z_1, z_2) (together with their errors) as calculated from Eq. (11) for the up quarks (marked U) and the down quarks (marked D). For comparison with the lepton case the position of δ_L is also shown (marked with a dot L). Slanted solid lines represent constant δ ($= 0, 1/27, 2/27, \dots, 2/9, \dots$). It can be seen that the observed value of δ_U is consistent with $\delta_U = \delta_L/3 = 2/27$. Thanks to a huge top quark mass, the errors are quite small here. For the down sector the mass hierarchy is not as strong as in the up sector, and therefore the corresponding errors are much larger.

We have to remember, however, that at $\mu = M_Z$ both k_L and δ_L deviate from their 'perfect' values of 1 and $2/9$, by about 0.2 % and 0.5 % respectively (the

¹Hence, due to the above mentioned incompatibility, Rosen cannot simultaneously describe the electron and muon masses with maximal precision as required by the data.

Figure 1: Contour plot of $\delta_f(z_1, z_2)$ and the relevant points corresponding to the charged lepton (L), up (U), and down (D) quark sectors, as explained in the text.



deviation from $2/9$ is virtually indiscernible in the scale used for the presentation of Fig. 1). Apparently, we should be interested in the values of current quark masses at the low energy scale and not at $\mu = 2$ GeV as in Eq. (11). For the up quark sector, thanks to the huge mass of the top quark, such a change cannot significantly modify the position of the corresponding U point in Fig. 1. This is not the case for the down quarks. In order to show what happens there, we assumed that $\kappa = m_s/m_d$ is fixed at the value of $\kappa = 20.4$ as obtained at $\mu = 2$ GeV from Eq. (11) and as also valid at low energies when extracted from π and K masses (see e.g. [13]). The corresponding relation between z_1 and z_2 is marked in Fig. 1 with a dashed line. Two points along this line, corresponding to $m_s = 130$ and 160 MeV, are shown there as well. We observe that at the expected low-energy-scale value of the strange quark mass (i.e. for m_s around 160 MeV) the value of δ_D appears to be close to $2\delta_L/3 = 4/27$. The obtained low-energy-scale values of δ_U , δ_D and δ_L are therefore suggestive of a nice (even if only fairly approximate) symmetry between the lepton and quark sectors, with the values of $I_3 = -1/2$ particle phases δ_L, δ_D depending on weak hypercharge Y and given by

$$\delta(I_3 = -1/2, Y) = \frac{1}{9}(1 + |Y|), \quad (13)$$

and the $I_3 = +1/2$ particle phases δ_ν, δ_U expected to be given by

$$\delta(I_3 = +1/2, Y) = \frac{1}{9}(1 - |Y|), \quad (14)$$

together forming an equally-spaced set and satisfying a lepton-quark sum rule $\delta_L + \delta_\nu = \delta_U + \delta_D$.

Obviously, due to the possible Majorana mass term, the observed masses of neutrinos do not have to realize the pattern $m_1 = m_2 < m_3$ suggested by $\delta(I_3 = +1/2, |Y| = 1) = 0$.

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